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Expectation values of local fields in Bullough-Dodd model and integrable perturbed conformal field theories

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Abstract

Exact expectation values of the fields $e^{a\varphi}$ in the Bullough-Dodd model are derived by adopting the "reflection relations" which involve the reflection S-matrix of the Liouville theory, as well as special analyticity assumption. Using this result we propose explicit expressions for expectation values of all primary operators in the $c < 1$ minimal CFT perturbed by the operator $\Phi_{1,2}$ or $\Phi_{2,1}$. Some results concerning the $\Phi_{1,5}$ perturbed minimal models are also presented.

1. Introduction

Computation of vacuum expectation values (VEV) of local fields (or one-point correlation functions) is important problem of quantum field theory (QFT) [1], [2]. When applied to statistical mechanics the VEV determine “generalized susceptibilities”, i.e. linear response of the system to external fields. More importantly, in QFT defined as a perturbed conformal field theory the VEV provide all information about its correlation functions which is not accessible through straightforward calculations in conformal perturbation theory [3]. Recently some progress was made in calculation of the VEV in 1+1 dimensional integrable QFT. In [4] explicit expression for the VEV of exponential fields in the sine-Gordon and sinh-Gordon models was proposed. It was found in [5] that this expression can be obtained as minimal solution to certain “reflection relations” which involve the Liouville “reflection S-matrix” [6], provided one assumes simple analytic properties of the VEV. This result for the sine-Gordon model allows one to obtain, through the quantum group restriction, expectation values of primary fields in $c < 1$ minimal CFT perturbed by the operator $\Phi_{1,3}$, with good agreement with numerical data [7]. In this paper we use the “reflection relations” to obtain the VEV of exponential fields $e^{a\varphi}$ in the so called Bullough-Dodd model [8], [9]. As is known [10], for special pure imaginary values of its coupling constant the Bullough-Dodd model admits quantum group restriction leading to a $c < 1$ minimal conformal field theories (CFT) perturbed by the operator $\Phi_{1,2}$. We use this relation to obtain the VEV of primary fields in these perturbed minimal CFT.

In Sect.2 we present some details of the derivation of the VEV in the sinh-Gordon and sine-Gordon models using the “reflection relations”, and show how the VEV of primary fields in minimal CFT perturbed by $\Phi_{1,3}$ can be obtained. In Sect.3 we extend this approach and find explicit expression for the VEV of the exponential fields $e^{a\varphi}$ in the Bullough-Dodd model. We show that in the semi-classical limit our expression agrees with known results from the classical Bullough-Dodd theory. We also run some perturbative checks. In Sect.4 we study the minimal CFT perturbed by the operator $\Phi_{1,2}$. Using our result for the Bullough-Dodd model we propose exact formula for the VEV of all primary fields $\Phi_{l,k}$ in these perturbed theories. We also compare our results with numerical data available in literature. In Sects.5 and 6 some results and conjectures concerning minimal models perturbed by the operators $\Phi_{1,5}$ and $\Phi_{2,1}$ are presented.

2. Reflection relations in the sinh-Gordon model

The sinh-Gordon model is defined by the Euclidean action

$$\mathcal{A}_{shG} = \int d^2x \left\{ \frac{1}{16\pi}(\partial_\nu\varphi)^2 + \mu e^{b\varphi} + \mu e^{-b\varphi} \right\} . \tag{2.1}$$

We are interested in the expectation values of the exponential fields,

$$G(a) = \langle\, e^{a\varphi} \,\rangle_{shG} \, . \tag{2.2}$$

As was observed in [5] these expectation values satisfy the "reflection relation"

$$G(a) = R(a)\, G(Q-a)\, , \tag{2.3}$$

$$G(-a) = R(a)\, G(-Q+a)\, , \tag{2.4}$$

where

$$Q = b^{-1} + b \tag{2.5}$$

and the function R is related to the so called Liouville reflection amplitude S [6],

$$R\big(\frac{Q}{2} + iP\big) = S(P) = -\left(\frac{\pi\mu\,\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{-\frac{2iP}{b}} \frac{\Gamma\big(1+2iP/b\big)\,\Gamma\big(1+2iPb\big)}{\Gamma\big(1-2iP/b\big)\,\Gamma\big(1-2iPb\big)} \, . \tag{2.6}$$

Note that the second of the relations (2.4) follows from the first one if one takes into account an obvious symmetry of (2.2),

$$G(a) = G(-a) \, .$$

No rigorous proof of the reflection relations (2.3), (2.4) is known to us. Here we give simple intuitive argument in support of these relations.

Let us note that the sinh-Gordon theory (2.1) can be interpreted as the perturbed Liouville QFT in two different ways. First, one could take the first two terms in the action (2.1) as the action $\mathcal{A}_L$ of the Liouville theory (in a flat 2D background metric) and treat the last term containing $e^{-b\varphi}$ as the perturbation. Then naively one could write down the conformal perturbation theory series (expansion in the perturbation term) for the one-point function of (2.1),

$$\langle\, e^{a\varphi}(x) \,\rangle_{shG} = Z^{-1} \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n!} \int d^2y_1 \ldots d^2y_n \,\langle\, e^{a\varphi}(x) e^{-b\varphi}(y_1) \ldots e^{-b\varphi}(y_n) \,\rangle_L \ , \tag{2.7}$$

where $\langle \ldots \rangle_L$ are the expectation values over the Lioville theory $\mathcal{A}_L$, and Z is its partition function which does not depend on a. With this expression the first "reflection relation" (2.3) follows from the reflection property of the Liouville correlation functions (see [6] for the details),

$$\langle\, e^{a\varphi}(x)\ldots\rangle_L = R(a)\,\langle\, e^{(Q-a)\varphi}(x)\ldots\rangle_L \;, \tag{2.8}$$

where dots stand for any local insertions. The coefficient function $R(a)$ is related to the Liouville two-point correlation function

$$\langle\, e^{a\varphi}(x)e^{a'\varphi}(x')\,\rangle_L = \big[\,\delta_{Q-a,a'} + R(a)\,\delta_{a,a'}\,\big]\;|x-x'|^{-4a(Q-a)}\;; \tag{2.9}$$

its explicit form is given by (2.6). The function S in (2.6) can be interpreted as the amplitude of scattering off the "Liouville wall", as explained in [6]. Alternatively, one could interpret the second term in (2.1) as the perturbation of the Liouville CFT defined by the first and the third terms in (2.1). Then writing down corresponding naive conformal perturbation theory series analogous to (2.7) one would arrive at the second relation (2.4). In both cases the problem is that the integrals in (2.7) (as well as the integrals appearing with the second interpretation) are highly infrared divergent and therefore the naive series (2.7) does not give a viable definition of the one-point function.

One can get around the above infrared problem as follows. Consider 2D "world sheet" Σ_g, topologically a sphere, equipped with the metric $g_{\nu\sigma}(x)$, and define a version of the sinh-Gordon theory on Σ_g with the following non-minimal coupling to the background metric g,

$$\mathcal{A}^g_{shG} = \int d^2x\sqrt{g}\,\Big\{\,\frac{1}{16\pi}\,g^{\nu\sigma}\partial_\nu\varphi\partial_\sigma\varphi + \frac{Q\hat{R}}{8\pi}\,\varphi\, + \mu : e^{b\varphi} :_g + \mu : e^{-b\varphi} :_g\,\Big\}\;, \tag{2.10}$$

where Q is given by (2.5), $\hat{R}$ denotes the scalar curvature of g and the symbol $: e^{\pm b\varphi} :_g$ signifies that these exponential fields are renormalized with respect to the background metric g. The first three terms in (2.10) define conformaly invariant Liouville theory $\mathcal{A}^g_L$ on Σ_g so that (2.10) agrees with the first of the above interpretations of the sinh-Gordon model as the perturbed Liouville theory $\mathcal{A}_L$. Precisely this was our reason for adding the curvature term in (2.10). Due to its conformal invariance the Liouville theory is insensitive to a choice of the background metric g. If one picks a conformal coordinates on Σ_g, so that

$$g_{\nu\sigma}(x) = \rho(x)\,\delta_{\nu\sigma}\;, \tag{2.11}$$

the dependence on $\rho(x)$ can be expelled from the Liouville part of the action (2.10) by the shift

$$\varphi(x) \to \varphi(x) - Q \log \rho(x) \ . \tag{2.12}$$

This transformation brings (2.10) to the form (up to field independent constant)

$$\mathcal{A}^g_{shG} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\nu \varphi)^2 + \mu e^{b\varphi} + \mu \rho^{2+2b^2} e^{-b\varphi} \right\} + Q\varphi_\infty \ , \tag{2.13}$$

where now the exponential fields $e^{\pm b\varphi} \equiv : e^{\pm b\varphi} :_{g^{(0)}}$ are normalized with respect to the flat metric $g^{(0)}_{\nu\sigma} = \delta_{\nu\sigma}$, so that

$$e^{\pm b\varphi}(x) = [\rho(x)]^{-b^2} : e^{\pm b\varphi} :_g \ .$$

The term with $\varphi_\infty = \lim_{|x|\to\infty} \varphi(x)$ plays no role in the perturbed theory. To be definite, let us take the metric g to be that of a sphere with area A,

$$\rho(x) = \left(1 + \pi |x|^2/A\right)^{-2} \ . \tag{2.14}$$

For finite A the conformal perturbation theory for $\langle\, e^{a\varphi} \,\rangle^g_{shG}$ in (2.13) (the expansion in $e^{-b\varphi}$) makes much better sense because now the integrals analogous to those in (2.7) contain the factors $\prod_{k=1}^n \big[\, \rho(y_k) \,\big]^{2+2b^2}$ providing an efficient infrared cutoff. As the result these calculations produce a power series of the form

$$\langle\, e^{a\varphi} \,\rangle^g_{shG} = \mu^{-\frac{a}{b}} \, A^{a(a-Q)} \sum_{n=2}^{\infty} \big[\, \mu A^{1+b^2} \,\big]^{2n} \, G_n(a) \ . \tag{2.15}$$

Owing to the property (2.8) of the Liouville correlation function each term in (2.15) satisfies the reflection relation (2.3). Assuming that the series (2.15) defines a function $G(a,t) = \sum_{n=2}^\infty t^n \, G_n(a)$ with the asymptotic

$$G(a,t) \to \ G(a) \ t^{\frac{a}{2b}(1-\frac{a}{Q})} \qquad \text{as} \ \ t \to \infty \tag{2.16}$$

and taking the limit $A \to \infty$ (which brings (2.10) back to (2.1)) one arrives at (2.3). Similarly, starting with the action which differs from (2.10) only in the sign of the curvature term, one can repeat the above arguments, this time taking the term with $e^{-b\varphi}$ as the perturbation. This leads to (2.4).

Of course, these arguments do not give a rigorous proof of the reflection relation (2.3) because the convergence of the conformal perturbation theory (2.15) is not at all

obvious even at finite A. Also, the existence of appropriate limiting behavior $t \to \infty$ in (2.16) is at least problematic. On the other hand if one *assumes* these arguments valid the reflection relations (2.3), (2.4) can be taken as the starting point in *deriving* the expectation values (2.2). In fact, if nothing is said about analytic properties of the function $G(a)$ the equations (2.3), (2.4) are not nearly sufficient to determine it. However, if one makes additional assumption that $G(a)$ is a *meromorphic* function of a, the following "minimal solution" to the equations (2.3), (2.4) is readily derived

$$\langle\, e^{a\varphi}\,\rangle_{shG} = \left[\frac{m\,\Gamma\big(\frac{1}{2+2b^2}\big)\,\Gamma\big(1+\frac{b^2}{2+2b^2}\big)}{4\sqrt{\pi}}\right]^{-2a^2} \times \\ \exp\left\{\int_0^\infty \frac{dt}{t}\left[-\frac{\sinh^2(2abt)}{2\sinh(b^2t)\,\sinh(t)\,\cosh\big((1+b^2)t\big)} + 2a^2\,e^{-2t}\right]\right\}\,, \tag{2.17}$$

where [11]

$$m = \frac{4\sqrt{\pi}}{\Gamma\big(\frac{1}{2+2b^2}\big)\,\Gamma\big(1+\frac{b^2}{2+2b^2}\big)}\,\left[-\frac{\mu\pi\Gamma\big(1+b^2\big)}{\Gamma\big(-b^2\big)}\right]^{\frac{1}{2+2b^2}} \tag{2.18}$$

is the particle mass of the sinh-Gordon model. Note that (2.17) is exactly the expression conjectured in [4]. At the moment we have absolutely no clue on how to justify this analyticity assumption. We can only make a remark that while the above arguments leading to the reflection relations (2.3), (2.4) do not seem to depend on the integrability of the sinh-Gordon model, the simple analytic properties assumed above most likely do[1]. However, we consider various perturbative checks of (2.17) performed in [4] as a strong evidence supporting both the above arguments about the reflection relations and the analyticity assumption. Additional support is provided by the results in Ref. [5] where these assumptions are used to derive the expectation values of the boundary operators in boundary sine-Gordon model with zero bulk mass. Furthermore, in Sect.3 we will use the same assumptions to obtain the expectation values of exponential fields in the Bullough-Dodd model.

As mentioned in [4], the expression (2.17) can be used to obtain the expectation values $\langle\, \Phi_{l,k}\,\rangle$ of primary fields with conformal dimensions

$$\Delta_{l,k} = \frac{(p'l-pk)^2-(p'-p)^2}{4pp'} \tag{2.19}$$

[1] This can be compared with the situation in lattice models of statistical mechanics. While so called "inversion relations" (see [12]) for the partition function can be written down for many models including non-integrable ones, only integrable lattice models provide enough analyticity for making the inversion relations a powerful tool of computing the partition functions.

in perturbed "minimal models" [13]

$$\mathcal{M}_{p/p'} + \lambda \int d^2x \, \Phi_{1,3}(x) \ . \tag{2.20}$$

This is possible because the perturbed minimal models (2.20) can be understood in terms of "quantum group restriction" of the sine-Gordon model [14], [15], [16], [17]

$$\mathcal{A}_{sG} = \int d^2x \left\{ \frac{1}{16\pi}(\partial_\nu \varphi)^2 - 2\mu \cos \beta\varphi \right\} . \tag{2.21}$$

As is known the sine-Gordon theory in infinite space-time exhibits a symmetry with respect to affine quantum group $U_q(\hat{sl}_2)$ with the level equal to zero and

$$q = e^{\frac{i\pi}{\beta^2}} \ . \tag{2.22}$$

The soliton-antisoliton doublet transforms as two-dimensional irreducible representation while the bound states are scalars. The S-matrix commutes with the generators $E_\pm, H_\pm, F_\pm$ which satisfy the relations

$$[E_+, F_+] = \frac{q^{H_+} - q^{-H_+}}{q - q^{-1}} \ ; \quad [E_-, F_-] = \frac{q^{H_-} - q^{-H_-}}{q - q^{-1}} \ ; \quad H_+ + H_- = 0 \ . \tag{2.23}$$

The operator H_+ is identified with the soliton charge. Important observation made in [16], [17] is that special exponential fields

$$V_{1,k}(x) = e^{i\frac{1-k}{2}\beta\varphi}(x) \qquad (k = 1, 2, ...) \tag{2.24}$$

commute with the generators E_+, H_+, F_+ of the subalgebra $U_q(sl_2)_+ \in U_q(\hat{sl}_2)$,

$$[E_+, V_{1,k}(x)] = [H_+, V_{1,k}(x)] = [F_+, V_{1,k}(x)] = 0 \ . \tag{2.25}$$

This subalgebra plays the central role in the relation between (2.21) and (2.20). The space of states $\mathcal{H}_{sG}$ of the sine-Gordon model admits special inner product [16] (different from the standard sine-Gordon scalar product) such that $E_+^\dagger = F_+$ (the standard scalar product implies $E_+^\dagger = F_-$). If q is a root of 1, i.e.

$$\beta^2 = \frac{p}{p'} \ , \tag{2.26}$$

where p, p' are relatively prime integers such that $p' > p > 1$, one can isolate the subspace $\mathcal{H}_p \in \mathcal{H}_{sG}$ consisting of the representations of $U_q(sl_2)_+$ with the spins $j =$

$0, 1/2, 1, \dots, p/2-1$. The space $Inv(\mathcal{H}_p)$ of invariant tensors of $\mathcal{H}_p$ is identified with the space of states of the perturbed minimal model (2.20). The relation between the sine-Gordon parameter μ and the coupling constant λ in (2.20) is found in [11],

$$\lambda = \frac{\pi\,\mu^2}{(1-2\beta^2)(3\beta^2-1)}\left[\frac{\Gamma^3(1-\beta^2)\,\Gamma(3\beta^2)}{\Gamma^3(\beta^2)\,\Gamma(1-3\beta^2)}\right]^{\frac{1}{2}} . \tag{2.27}$$

The theory (2.20) has $p-1$ degenerate ground states $\mid 0_s\rangle$, $s = 1, 2, \dots, p-1$ [18] which can be associated with the nodes of the Dynkin diagram A_{p-1}. The excitations are the kinks interpolating between these vacua, and possibly some neutral particles interpreted as bound states of the kinks. According to (2.25) the operators (2.24) of the sine-Gordon model are related in a simple way to the primary fields $\Phi_{1,k}$ of (2.20)

$$V_{1,k}(x) = N_{1,k}\ \Phi_{1,k}(x)\ , \tag{2.28}$$

where $N_{1,k}$ are numerical factors which depend on the normalization of $\Phi_{1,k}$. The canonical normalization

$$\langle\, \Phi_{1,k}(x)\Phi_{1,k}(x')\,\rangle \to |x-x'|^{-4\Delta_{1,k}} \qquad \text{as} \qquad |x-x'| \to 0 \tag{2.29}$$

corresponds to the choice

$$N_{1,k}^2 = \mathcal{R}\big(\,\frac{1-k}{2}\,\beta\,\big)/\mathcal{R}(0)\ , \tag{2.30}$$

where

$$\mathcal{R}(\alpha) = -\Big(-\frac{\pi\mu\,\Gamma(-\beta^2)}{\Gamma(1+\beta^2)}\Big)^{1-\frac{1}{\beta^2}-\frac{2\alpha}{\beta}}\ \frac{\Gamma\big(\beta^{-2}+2\alpha\beta^{-1}\big)\,\Gamma\big(\beta^2-2\alpha\beta\big)}{\Gamma\big(2-\beta^{-2}-2\alpha\beta^{-1}\big)\,\Gamma\big(2-\beta^2+2\alpha\beta\big)}\ .$$

is obtained from (2.6) by the substitution

$$b \to i\beta\,, \quad a \to i\alpha\,, \quad \mu \to -\mu\ . \tag{2.31}$$

Notice that the relation (2.27) can be written as $\lambda = -N_{1,3}\,\mu$. With the normalization (2.29), one finds

$$\langle 0_s \mid \Phi_{1,k} \mid 0_s\rangle = (-1)^{s(k-1)}\ N_{1,k}\ \mathcal{G}\big(\,\frac{1-k}{2}\,\beta\,\big)\ , \tag{2.32}$$

where $\mathcal{G}(\alpha)$ is related to $G(a)$ in (2.17) by the same substitution (2.31). The sign factor in (2.32) takes into account the fact that the exponential fields (2.24) with even k change sign when φ is translated by the period of the potential term in (2.21).

For the primary fields $\Phi_{l,k}$ with $l>1$ of the restricted theory (2.20) the situation is more difficult. The exponential fields

$$\exp\left\{i\Big(\frac{l-1}{2\beta}-\frac{k-1}{2}\beta\Big)\,\varphi\right\}$$

for $l>1$ are not invariant with respect to the algebra $U_q(sl_2)_+$. Together with certain nonlocal fields they form finite-dimensional representations of this algebra. The calculations become much more involved and we did not complete them yet. However, we have a conjecture

$$\langle 0_s\mid \Phi_{l,k}\mid 0_s\rangle=\frac{\sin\big(\frac{\pi s}{p}|p'l-pk|\big)}{\sin\big(\frac{\pi s}{p}(p'-p)\big)}\ \left[M\ \frac{\sqrt{\pi}\,\Gamma\big(\frac{3}{2}+\frac{\xi}{2}\big)}{2\ \Gamma\big(\frac{\xi}{2}\big)}\right]^{2\Delta_{l,k}}\ \mathcal{Q}_{1,3}\big(\,(\xi+1)l-\xi k\,\big)\ , \qquad (2.33)$$

where [11]

$$M=\frac{2\,\Gamma\big(\frac{\xi}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{1}{2}+\frac{\xi}{2}\big)}\left[\frac{\pi\,\lambda\,(1-\xi)(2\xi-1)}{(1+\xi)^2}\ \sqrt{\frac{\Gamma\big(\frac{1}{1+\xi}\big)\,\Gamma\big(\frac{1-2\xi}{1+\xi}\big)}{\Gamma\big(\frac{\xi}{1+\xi}\big)\,\Gamma\big(\frac{3\xi}{1+\xi}\big)}}\ \right]^{\frac{1+\xi}{4}} \qquad (2.34)$$

is the kink mass and

$$\xi=\frac{p}{p'-p}\ . \qquad (2.35)$$

The function $\mathcal{Q}_{1,3}(\eta)$ for $\big|\Re e\ \eta\big|<\xi$ in (2.33) is given by the integral

$$\mathcal{Q}_{1,3}(\eta)=\exp\left\{\int_0^\infty\frac{dt}{t}\ \left(\frac{\cosh(2t)\ \sinh\big(t(\eta-1)\big)\ \sinh\big(t(\eta+1)\big)}{2\cosh(t)\sinh(t\xi)\sinh\big(t(1+\xi)\big)}-\frac{(\eta^2-1)}{2\xi(\xi+1)}\,e^{-4t}\right)\right\}$$

and it is defined by analytic continuation outside this domain [2]. In writing (2.33) we assumed the same canonical normalization convention for the fields $\Phi_{l,k}$ as in (2.29), i.e.

$$\langle 0_s\mid\ \Phi_{l,k}(x)\Phi_{l,k}(x')\ \mid 0_s\rangle\to|x-x'|^{-4\Delta_{l,k}}\qquad \text{as}\qquad |x-x'|\to 0\ . \qquad (2.36)$$

Notice that (2.33) automatically satisfy the relation

$$\langle 0_s\mid\Phi_{l,k}\mid 0_s\rangle=\langle 0_s\mid\Phi_{p-l,p'-k}\mid 0_s\rangle\ .$$

[2] The expression for $\langle\,\Phi_{l,k}\,\rangle$ proposed in [4] does not contain the first factor (2.33) which carries the dependence on s. However, the formula in [4] is equivalent to (2.33) if the expectation values in [4] are understood not as the matrix elements between the above ground states $\mid 0_s\rangle$, but rather as the matrix elements between certain superpositions of these states which arise in the limit $L\to\infty$ from the asymptotically degenerate states of the finite-size system, with the spatial coordinate compactified on a circle of circumference L. We will explain this point elsewhere.

3. Vacuum expectation values in the Bullough-Dodd model

The Bullough-Dodd model is defined by the action [8], [9]

$$\mathcal{A}_{BD}=\int d^2x\Big\{\frac{1}{16\pi}(\partial_\nu\varphi)^2+\mu e^{b\varphi}+\mu' e^{-\frac{b}{2}\varphi}\Big\}\ . \tag{3.1}$$

There is some redundancy in having two parameters μ and μ' in (3.1) because if one shifts the field variable in (3.1),

$$\varphi\to\varphi+\varphi_0\ , \tag{3.2}$$

they change as $\mu\to\mu e^{b\varphi_0}\ ,\ \mu'\to\mu' e^{-\frac{b}{2}\varphi_0}$, so that only the combination $\mu\,(\mu')^2$ is invariant. Nonetheless, we will keep both parameters. In fact in what follows the combination

$$m=\frac{2\sqrt{3}\ \Gamma\big(\frac{1}{3}\big)}{\Gamma\big(1+\frac{b^2}{6+3b^2}\big)\,\Gamma\big(\frac{2}{6+3b^2}\big)}\ \bigg[-\frac{\mu\pi\,\Gamma\big(1+b^2\big)}{\Gamma\big(-b^2\big)}\bigg]^{\frac{1}{6+3b^2}}\ \bigg[-\frac{2\mu'\pi\,\Gamma\big(1+\frac{b^2}{4}\big)}{\Gamma\big(-\frac{b^2}{4}\big)}\bigg]^{\frac{2}{6+3b^2}} \tag{3.3}$$

is proven to be useful. Note that m is invariant under the shift (3.2). The model (3.1) is integrable and its factorizable S-matrix is described in [19]. It contains a single neutral particle. We will show below that the mass of this particle coincides with the parameter m defined in (3.3). In this and the subsequent sections we use the notation

$$G_{BD}(a)=\langle\, e^{a\varphi}\,\rangle_{BD} \tag{3.4}$$

for the expectation value in the Bullough-Dodd model, where the exponential field is assumed to be normalized in accordance with the following short distance operator product expansion,

$$e^{a\varphi}(x)\,e^{a'\varphi}(x')\to|x-x'|^{-4aa'}\ e^{(a+a')\varphi}(x')\qquad\text{as}\qquad|x-x'|\to 0\ . \tag{3.5}$$

Notice that $|a|$ and $|a'|$ should be sufficiently small numbers, in order for (3.5) to be a leading asymptotic.

Exactly as in the case of the sinh-Gordon model in Sect.2, the Bullough-Dodd model can be interpreted as the perturbed Liouville theory in two different ways, with either $e^{b\varphi}$ or $e^{-\frac{b}{2}\varphi}$ taken as the perturbing operator. Using these interpretations and repeating the arguments in Sect.2 which led to (2.3), (2.4), one arrives at two reflection relations for (3.4)

$$\begin{aligned}G_{BD}(a)&=R(a)\,G_{BD}(Q-a)\,,\\ G_{BD}(-a)&=R'(a)\,G_{BD}(-Q'+a)\end{aligned} \tag{3.6}$$

with

$$Q = \frac{1}{b} + b\,, \qquad Q' = \frac{2}{b} + \frac{b}{2}\ . \tag{3.7}$$

The function $R(a)$ is exactly the same as in (2.6), while $R'(a)$ is obtained from that by the substitution $\mu \to \mu'$, $b \to b/2$. As in Sect.2, let us now assume that $G_{BD}(a)$ is a meromorphic function of a. Then the following minimal solution to the equations (3.6) is immediately obtained

$$\begin{aligned}\langle\, e^{a\varphi}\,\rangle_{BD} =& \left[\frac{\mu'}{\mu}\ \frac{2^{\frac{b^2}{2}}\,\Gamma(1-b^2)\,\Gamma\big(1+\frac{b^2}{4}\big)}{\Gamma(1+b^2)\,\Gamma\big(1-\frac{b^2}{4}\big)}\right]^{\frac{2a}{3b}}\ \left[\frac{m\,\Gamma\big(1+\frac{b^2}{6+3b^2}\big)\,\Gamma\big(\frac{2}{6+3b^2}\big)}{2^{\frac{2}{3}}\,\sqrt{3}\,\Gamma\big(\frac{1}{3}\big)}\right]^{ab-2a^2}\times\\ &\exp\left\{\int_0^{+\infty}\frac{dt}{t}\,\left(-\frac{\sinh\big((2+b^2)t\big)\,\Psi(t,a)}{\sinh\big(3(2+b^2)t\big)\,\sinh(2t)\,\sinh\big(b^2t\big)}+2a^2\,e^{-2t}\right)\right\}\,,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}\Psi(t,a) =& \sinh\big(2abt\big)\Big(\,\sinh\big((4+b^2+2ab)t\big)-\sinh\big((2+2b^2-2ab)t\big)+\\ &\sinh\big((2+b^2+2ab)t\big)-\sinh\big((2+b^2-2ab)t\big)-\sinh\big((2-b^2+2ab)t\big)\Big)\ .\end{aligned}$$

and m is given by (3.3). The integral in (3.8) is convergent if

$$-\frac{1}{b}-\frac{b}{4}<\Re e\, a<\frac{1}{2b}+\frac{b}{2}\ ;$$

it should be understood in terms of analytic continuation otherwise. We propose (3.8) as exact expectation values for the Bullough-Dodd model (3.1).

Expanding $\langle\, e^{a\varphi}\,\rangle_{BD} = 1 + a\,\langle\,\varphi\,\rangle_{BD} + O(a^2)$, one finds the expectation value of φ,

$$\begin{aligned}\langle\,\varphi\,\rangle_{BD} =& \frac{2}{3b}\ \log\left\{\frac{\mu'}{\mu}\ \frac{\Gamma(1-b^2)\,\Gamma\big(1+\frac{b^2}{4}\big)}{\Gamma(1+b^2)\,\Gamma\big(1-\frac{b^2}{4}\big)}\ \left[\frac{m\,\Gamma\big(1+\frac{b^2}{6+3b^2}\big)\,\Gamma\big(\frac{2}{6+3b^2}\big)}{2^{\frac{1}{3}}\,\sqrt{3}\,\Gamma\big(\frac{1}{3}\big)}\right]^{\frac{3}{2}b^2}\right\}-\\ 8b&\int_0^{+\infty}dt\,\frac{\sinh\big((2+b^2)t\big)\,\sinh\big((1-\frac{b^2}{2})t\big)\,\sinh\big((2+\frac{b^2}{2})t\big)\,\sinh\big((1+b^2)t\big)}{\sinh\big(3(2+b^2)t\big)\,\sinh(2t)\,\sinh\big(b^2t\big)}\ .\end{aligned} \tag{3.9}$$

For $a=b$ and for $a=-b/2$ the integral in (3.8) can be evaluated explicitly,

$$\mu\,\langle\, e^{b\varphi}\,\rangle_{BD} = \mu'/2\ \langle\, e^{-\frac{b}{2}\varphi}\,\rangle_{BD} = \frac{m^2}{24\,\sqrt{3}\,(2+b^2)\,\sin\big(\frac{\pi b^2}{6+3b^2}\big)\,\sin\big(\frac{2\pi}{6+3b^2}\big)}\ . \tag{3.10}$$

These expectation values can be used to derive the bulk specific free energy of the Bullough-Dodd model

$$f_{BD} = -\lim_{V\to\infty}\frac{1}{V}\,\log Z_{BD}\ , \tag{3.11}$$

where V is the volume of the 2D space and Z_{BD} is the singular part of the partition function associated with (3.1). Obviously,

$$\partial_\mu f_{BD} = \langle\, e^{b\varphi}\,\rangle_{BD}\,; \qquad \partial_{\mu'} f_{BD} = \langle\, e^{-\frac{b}{2}\varphi}\,\rangle_{BD}\ . \tag{3.12}$$

This leads to the following result,

$$f_{BD} = \frac{m^2}{16\sqrt{3}\,\sin\big(\frac{\pi b^2}{6+3b^2}\big)\,\sin\big(\frac{2\pi}{6+3b^2}\big)}\ . \tag{3.13}$$

On the other hand exact expression for the specific free energy in terms of the physical particle mass can be obtained from exact form-factors [20], [21], or from the Thermodynamic Bethe Ansatz calculations following [22], [23]. This way one obtains exactly (3.13) with m understood as the particle mass. This shows that (3.3) indeed gives the particle mass in the Bullough-Dodd model.

Since the above derivation of (3.8) is based on the assumptions (in particular, we made strong analyticity assumption), some checks of this result are desirable. Simple consistency check is based on the known fact that for $b^2 = 2$ the Bullough-Dodd model is equivalent to the sinh-Gordon model (2.1) with $b^2 = 1/2$. It is possible to check that (3.8) calculated at $b^2 = 2$ coincides with (2.17) calculated at $b^2 = 1/2$.

Important check can be performed in the classical limit $b^2 \to 0$. Consider the expectation value $\langle\, e^{\frac{\sigma}{b}\varphi}\,\rangle_{BD}$. In the limit $b^2 \to 0$ with σ fixed, (3.8) gives

$$\log\langle\, e^{\frac{\sigma}{b}\varphi}\,\rangle_{BD} = \frac{2}{b^2}\bigg(-\sigma^2\,\log m + \frac{\sigma}{3}\,\log\Big(\frac{\mu'}{2\mu}\Big) + \int_0^\sigma d\omega\, C(\omega)\bigg) + O(1)\ , \tag{3.14}$$

where

$$C(\omega) = \big(\,2\log 2 + 3\log 3\,\big)\,\omega + \log\bigg[\,\frac{\Gamma\big(\frac{1+\omega}{3}\big)\,\Gamma\big(\frac{2+2\omega}{3}\big)}{\Gamma\big(\frac{2-\omega}{3}\big)\,\Gamma\big(\frac{1-2\omega}{3}\big)}\,\bigg]\ . \tag{3.15}$$

On the other hand, for $b^2 \to 0$ the expectation value $\langle\, e^{\frac{\sigma}{b}\varphi}(0)\,\rangle_{BD}$ can be calculated directly in terms of the action (3.1) evaluated on appropriate classical solution $\varphi_{cl}(x)$ of the equations of motion associated with (3.1). The suitable solution depends only on the radial coordinate $r = |x|$. It can be written as $\varphi_{cl}(r) = \frac{2}{b}\,\big(\phi(r) + \frac{1}{3}\log(\frac{\mu'}{2\mu})\,\big)$, where $\phi(r)$ is the solution to the classical Bullough-Dodd equation

$$\partial_r^2\phi + r^{-1}\partial_r\phi = \frac{m^2}{3}\,\Big(\,e^{2\phi} - e^{-\phi}\,\Big)\ , \tag{3.16}$$

which satisfies the following asymptotic conditions

$$\begin{aligned} &\phi(r) \to -2\sigma\, \log(mr) + \tilde{C}(\sigma) \qquad \text{as} \;\; r \to 0 \;, \\ &\phi(r) \to \frac{4\sqrt{3}}{\pi}\; \sin\big(\frac{\pi\sigma}{3}\big)\, \sin\big(\frac{\pi}{3}(1-\sigma)\big)\; K_0(mr) \qquad \text{as} \;\; r \to +\infty \;, \end{aligned} \tag{3.17}$$

where $K_0(t)$ is the MacDonald function. The constant term $\tilde{C}(\sigma)$ in (3.17) is not arbitrary but must be consistently determined from the equation (3.16). Exact result for $\tilde{C}(\sigma)$ found in [24] (see also [25]) shows that it coincides with $C(\sigma)$ defined by (3.15). Then calculation of the classical action gives the result identical to (3.14).

One can go beyond the classical limit and consider the loop expansion for the expectation values (3.4). The simplest thing to study is the expectation value $\langle\, \varphi\, \rangle_{BD}$. According to (3.9), this quantity can be written as a power series in b,

$$\langle\, \varphi\, \rangle_{BD} = \frac{2}{3b}\, \log\Big(\, \frac{\mu'}{2\mu}\,\Big) + b\,\big(\,\gamma + \log(m/2)\,\big) + \frac{b^3}{108}\,\big(\,3\sqrt{3}\,\pi + 10\pi^2 - 15\,\psi'\big(2/3\big)\,\big) + O(b^5)\,, \tag{3.18}$$

where $\psi'(t) = \partial_t^2 \log\big(\Gamma(t)\big)$ and $\gamma = 0.577216...$ is Euler's constant. On the other hand, it is possible to calculate this expectation value within the standard Feynman perturbation theory for the action (3.1). The first term in (3.18) coincides with the classical value of φ, and the diagrams contributing to the next two orders are shown in Fig.1. Calculations are straightforward and the result is in exact agreement with (3.18).

4. Expectation values of primary fields in the minimal models perturbed by the operator $\Phi_{1,2}$

The expectation value (3.8) proposed in the previous section allows one to obtain the the expectation values

$$\mathcal{G}_{cBD}(\alpha) = \langle\, e^{i\alpha\varphi}\, \rangle_{cBD} \tag{4.1}$$

in the so called “complex Bullough-Dodd model”,

$$\mathcal{A}_{cBD} = \int d^2x \Big\{\, \frac{1}{16\pi}(\partial\varphi)^2 - \mu e^{i\beta\varphi} - \mu' e^{-i\frac{\beta}{2}\varphi}\, \Big\}\;, \tag{4.2}$$

which is obtained from (3.1) by replacing the parameter b by pure imaginary value $b \to i\beta$ and $\mu \to -\mu,\; \mu' \to -\mu'$. The action (4.2) is complex and it is not clear exactly how it defines a quantum field theory. Nonetheless, some formal manipulations can be

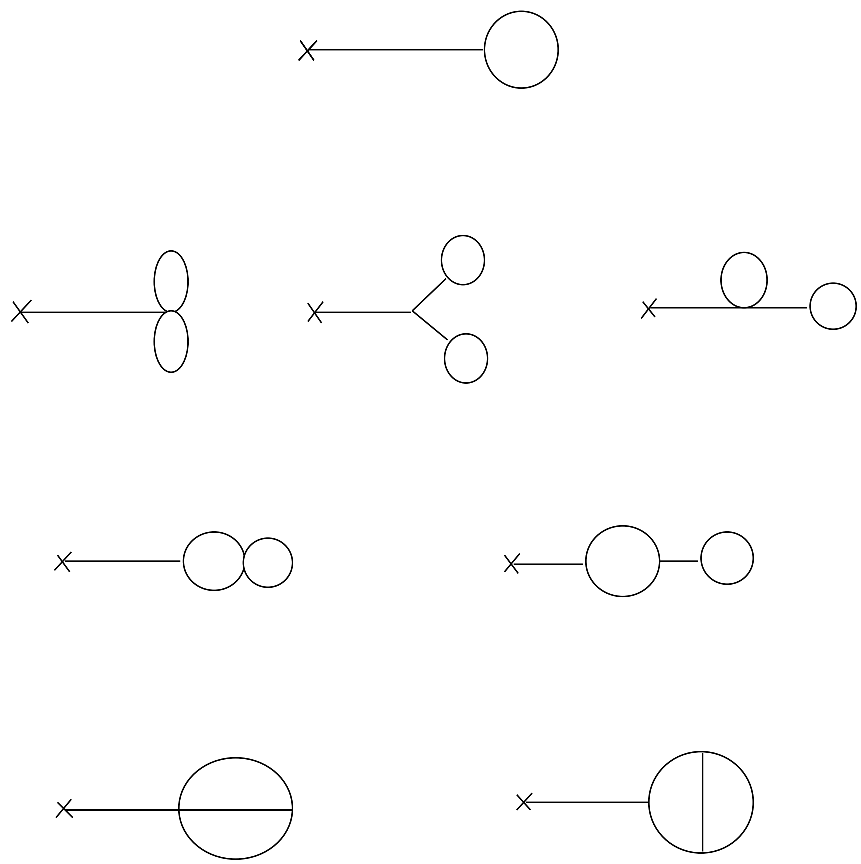

Fig.1. Feynman diagrams contributing to $\langle \varphi \rangle_{BD}$ to the orders b and b^3.

done. In particular, the model (4.2) is shown to be integrable and its factorizable S-matrix is identified in [10]. Although the model (4.2) is very different from (3.1) in its physical content (the model (4.2) contains solitons), there are good reasons to believe that the expectation values (4.1) are obtained from the expectation values (3.8) by simple substitution

$$\mathcal{G}_{cBD}(\alpha) = G_{BD}(i\alpha)\big|_{b=i\beta} \ . \tag{4.3}$$

The arguments are much the same as those which lead us to the relation between the expectation values in (2.1) and (2.21). Namely, it is easy to check that for fixed a the expression (3.8) can be expanded into a power series in b with a finite radius of convergence. This suggests that in principle (3.8) can be calculated by summing up the Feynman perturbation theory series for (3.1). At the same time the perturbation theory for (3.1) agrees with that for (4.2) to all orders if one makes the substitution $b \to i\beta$.

The complex Bullough-Dodd model admits the quantum group restriction similar to the one in the sine-Gordon model [10]. There are some differences, though. The model (4.2) (in the infinite space) has a symmetry with respect to the affine quantum group algebra $U_q(A_2^{(2)})$ where q is given by the same expression (2.22). This algebra contains a subalgebra $U_q(sl_2)$ which can be used for the quantum group restriction of (4.2)[3]. If β^2 takes the rational values (2.26) the restricted theory coincides with perturbed minimal model [10]

$$\mathcal{M}_{p/p'} + \lambda \int d^2x \, \Phi_{1,2}(x) \ . \tag{4.4}$$

with

$$\lambda^2 = -\frac{\pi \ \mu \, (\mu')^2}{(2\beta^2 - 1)^2} \, \frac{\Gamma^2(1-\beta^2)\,\Gamma(2\beta^2)}{\Gamma^2(\beta^2)\,\Gamma(1-2\beta^2)} \ .$$

As in (2.21), the generators E_+, H_+, F_+ of the subalgebra $U_q(sl_2)$ commute with the exponential fields (2.24) which become the primary fields $\Phi_{1,k}$ $(k = 1, 2, ..., p'-1)$ in the restricted theory (4.4). Therefore, the same arguments as in Sect.2 lead to the following expression for the expectation values of these primary fields in (4.4),

$$\langle 0_s \mid \Phi_{1,k} \mid 0_s \rangle = (-1)^{s(k-1)} \ N_{1,k} \ \mathcal{G}_{cBD}\big(\frac{1-k}{2} \, \beta \big) \ , \tag{4.5}$$

which is similar to (2.32), except that now $\mathcal{G}_{cBD}(\alpha)$ stands for the expectation value (4.1) in the complex Bullough-Dodd model. Here again, the canonical normalization (2.36) of the primary fields $\Phi_{l,k}$ is assumed. In (4.5) s is an integer which labels the ground states of the perturbed theory. Precisely which values it takes relates to the question of the vacuum structure of the perturbed theory (4.4); we will discuss this point a little later. For now, it suffices to note that only the signs of the the expectation values (4.5) depend on the choice of the vacuum.

Particular case of (4.5) is the expectation value of the perturbing operator,

$$\begin{aligned}\langle 0_s \mid \Phi_{1,2} \mid 0_s \rangle = (-1)^{s-1} \ |\lambda|^{\frac{\xi-2}{3\xi+6}} \ & \frac{2^{\frac{2\xi+10}{3\xi+6}}\,(\xi+1)\,\Gamma^2\big(\frac{1}{3}\big)}{\sqrt{3}\,(\xi+2)\,\pi^2} \, \frac{\Gamma\big(\frac{\xi+4}{3\xi+6}\big)\,\Gamma\big(\frac{\xi}{3\xi+6}\big)}{\Gamma\big(\frac{2\xi+2}{3\xi+6}\big)\,\Gamma\big(\frac{2\xi+6}{3\xi+6}\big)} \times \\ & \Bigg[\frac{\pi^2\,\Gamma^2\big(\frac{3\xi+4}{4\xi+4}\big)\,\Gamma\big(\frac{1}{2}+\frac{1}{\xi+1}\big)}{\Gamma^2\big(\frac{\xi}{4\xi+4}\big)\,\Gamma\big(\frac{1}{2}-\frac{1}{\xi+1}\big)} \Bigg]^{\frac{2\xi+2}{3\xi+6}} \ ,\end{aligned} \tag{4.6}$$

[3] The above affine quantum algebra contains another subalgebra, $U_{q^4}(sl_2)$, which can be used for another quantum group restriction of (4.2). We briefly discuss this case in Sect.5.

where ξ is given by (2.35). As usual, this expectation value is related to the specific free energy $f_{1,2}$ of the perturbed theory (4.4)

$$\partial_\lambda f_{1,2} = \langle 0_s \mid \Phi_{1,2} \mid 0_s \rangle \ . \tag{4.7}$$

On the other hand, the specific free energy for (4.4) is known exactly in terms of the mass M of the lightest kink present in this theory [23],

$$f_{1,2} = -\frac{M^2 \ \sin\left(\frac{\pi\,\xi}{3\xi+6}\right)}{4\sqrt{3}\,\sin\left(\frac{\pi(2\xi+2)}{3\xi+6}\right)} \ . \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), one finds the relation between the coupling constant λ in (4.4) and the mass M,

$$M = \frac{2^{\frac{\xi+5}{3\xi+6}}\,\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{\xi}{3\xi+6}\right)}{\pi\ \Gamma\left(\frac{2\xi+2}{3\xi+6}\right)}\ \left[\frac{\pi^2\,\lambda^2\,\Gamma^2\left(\frac{3\xi+4}{4\xi+4}\right)\Gamma\left(\frac{1}{2}+\frac{1}{\xi+1}\right)}{\Gamma^2\left(\frac{\xi}{4\xi+4}\right)\Gamma\left(\frac{1}{2}-\frac{1}{\xi+1}\right)}\right]^{\frac{\xi+1}{3\xi+6}}\ , \tag{4.9}$$

in exact agreement with [23]. According to (4.9), the perturbed QFT (4.4) develops a massive spectrum for

$$\begin{aligned} &0<\xi<1 \qquad && \Re e\,\lambda = 0\,;\\ &\xi>1\,, \qquad && \Im m\,\lambda = 0\,. \end{aligned} \tag{4.10}$$

In what follows we will discuss the second case only.

As in the case of $\Phi_{1,3}$ perturbation the situation with other primary fields $\Phi_{l,k}$ with $l>1$ in (4.4) is more difficult. However, there is a natural modification of the conjecture (2.32) suitable for (4.4),

$$\langle 0_s \mid \Phi_{l,k} \mid 0_s \rangle = \frac{\sin\left(\frac{\pi s}{p}|p'l-pk|\right)}{\sin\left(\frac{\pi s}{p}(p'-p)\right)}\ \left[\frac{M\,\pi\,(\xi+1)\,\Gamma\left(\frac{2\xi+2}{3\xi+6}\right)}{2^{\frac{2}{3}}\,\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{\xi}{3\xi+6}\right)}\right]^{2\Delta_{l,k}}\ \mathcal{Q}_{1,2}\big(\,(\xi+1)l-\xi k\,\big)\ , \tag{4.11}$$

where $\mathcal{Q}_{1,2}(\eta)$ for $\big|\Re e\ \eta\big|<\xi$ $(\xi>1)$ is given by the integral

$$\mathcal{Q}_{1,2}(\eta) = \exp\bigg\{\int_0^\infty \frac{dt}{t}\ \bigg(\frac{\sinh\big((\xi+2)t\big)\ \sinh\big(t(\eta-1)\big)\ \sinh\big(t(\eta+1)\big)}{\sinh\big(3(\xi+2)t\big)\sinh\big(2(\xi+1)t\big)\sinh\big(\xi t\big)}\times$$
$$\Big(\cosh\big(3(\xi+2)t\big)+\cosh\big((\xi+4)t\big)-\cosh\big((3\xi+4)t\big)+\cosh\big(\xi t\big)+1\Big)-$$
$$\frac{(\eta^2-1)}{2\xi(\xi+1)}\,e^{-4t}\bigg)\bigg\}\ ,$$

and is defined through analytic continuation outside this domain. The integer s in (4.11) labels the vacuum states of (4.4). To discuss the vacuum structure of (4.4) let us recall the basic idea of quantum group restriction. The states of (4.2) can be classified according to the representations of the above quantum algebra $U_q(sl_2)$. If β^2 takes the rational value (2.26) the subspace $\mathcal{H}_p \in \mathcal{H}_{cBD}$ consisting of the representations with the spins $j = 0, 1/2, 1, ..., p/2-1$, is closed with respect to the dynamics of (4.2). However unlike the sine-Gordon case, the solitons of (4.2) transform as the three-dimensional representations of $U_q(sl_2)$ with the spin $j = 1$. Therefore in fact there are two dynamically closed subspaces, $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$, containing respectively half-integer or integer spins out of the above set $j = 0, 1/2, ..., p/2 - 1$. Each of the spaces $Inv(\mathcal{H}_p^+)$ and $Inv(\mathcal{H}_p^-)$ of invariant tensors associated with $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$ can be interpreted as the space of states of certain quantum field theory. Therefore, this quantum group restriction of (3.1) gives rise to two different quantum field theories. Notice that if p is odd, the spaces $Inv(\mathcal{H}_p^+)$ and $Inv(\mathcal{H}_p^-)$ are isomorphic because of the known property of the tensor category of representations of $U_q(sl_2)$ with $q^p = \pm 1$, and hence the corresponding field theories are equivalent. If p is even, these are two really different field theories. Let us recall in this connection that the minimal CFT $\mathcal{M}_{p/p'}$ always has Z_2 symmetry which acts on the primary fields as

$$\Phi_{l,k} \to (-1)^{(l-1)p'-(k-1)p}\ \Phi_{l,k}\ . \tag{4.12}$$

If p is odd, the perturbing operator $\Phi_{1,2}$ is odd under the transformation (4.12), and therefore changing the sign of the perturbation in (4.4) leads to an equivalent field theory – all its correlation functions are obtained from those of the original theory by the substitution (4.12). On the contrast, if p is even, the operator $\Phi_{1,2}$ is invariant under (4.12); in this case the theories (4.4) with different signs of the perturbation are expected to be essentially different[4]. Therefore it is natural to identify $Inv(\mathcal{H}_p^+)$ with the space of states of (4.4) with $\lambda > 0$ and $Inv(\mathcal{H}_p^-)$ with the space of states of (4.4) with $\lambda < 0$. The integers s labeling the vacua in (4.11) are related to the spins j admitted into $\mathcal{H}_p$ as $s = 2j + 1$. We conclude that the theory (4.4) with odd p has $\frac{p-1}{2}$ degenerate ground states independently on the sign of λ; however, if $\lambda > 0$ these ground states are identified with the above vacua $\mid 0_s\rangle$ with even $s = 2, 4, ..., p-1$, while if $\lambda < 0$ these are the vacua $\mid 0_s\rangle$ with odd $s = 1, 3, ..., p-2$. If p is even and $\lambda > 0$ there are $p/2 - 1$ ground states identified with

[4] The exception is the case $(p, p') = (4, 5)$ where the theories with different signs of λ in (4.4) are related by duality transformation [26], [27].

$| 0_s\rangle$ with even $s = 2, 4, ..., p-2$. Finally, if p is even and $\lambda < 0$ the theory (4.4) has $p/2$ ground states $| 0_s\rangle$ with even $s = 1, 3, ..., p-1$. With this understanding, the formula (4.11) applies to all models (4.4), both with positive and negative λ.

In [7] the Truncated Conformal Space method [28] was adopted to obtain numerically the expectation values of primary fields in the perturbed theory (4.4) for some (p, p'). It is interesting to compare our results to these numerical data.

The model (4.4) with $(p, p') = (3, 4)$ describes the Ising model at critical temperature with nonzero magnetic field. In this case there is only one vacuum $|0\rangle \equiv |0_2\rangle$ (we assume that $\lambda > 0$), and (4.11) gives

$$\langle 0 \mid \Phi_{1,2} \mid 0\rangle = -1.27758...\ \lambda^{\frac{1}{15}}\ ,$$
$$\langle 0 \mid \Phi_{1,3} \mid 0\rangle = 2.00314...\ \lambda^{\frac{8}{15}}\ .$$

According to numerical calculations in this case

$$\langle 0 \mid \Phi_{1,2} \mid 0\rangle_{num} = -1.277(2)\ \lambda^{\frac{1}{15}}\ ,$$
$$\langle 0 \mid \Phi_{1,3} \mid 0\rangle_{num} = 1.94(6)\ \lambda^{\frac{8}{15}}\ .$$

The VEV $\langle 0 \mid \Phi_{1,3} \mid 0\rangle$ was also obtained from the fit of lattice data with the result $2.02(10)\ \lambda^{\frac{8}{15}}$ [7].

The case $(p, p') = (4, 5)$ in (4.4) describes Tricritical Ising model perturbed by the leading energy density operator $\varepsilon(x)$ of the conformal dimension $\Delta_{1,2} = \frac{1}{10}$. The minimal model $\mathcal{M}_{4/5}$ contains four more primary fields (besides the above field $\varepsilon = \Phi_{1,2}$ and the identity operator), the sub-leading energy density operator $\varepsilon' = \Phi_{1,3}$ with $\Delta_{1,3} = \frac{3}{5}$ (sometimes referred to as the “vacancy operator”), two magnetic operators $\sigma = \Phi_{2,2}$ ($\Delta_{2,2} = \frac{3}{80}$) and $\sigma' = \Phi_{2,1}$ ($\Delta_{2,1} = \frac{7}{16}$) and $\Phi_{1,4}$ (the latter does not have obvious physical interpretation). If $\lambda < 0$ the Ising symmetry $\sigma \to -\sigma$ is spontaneously broken and there are two ground states $| +\rangle \equiv | 0_1\rangle$ and $| -\rangle \equiv | 0_3\rangle$. If $\lambda > 0$ there is a unique ground state $| 0\rangle \equiv | 0_2\rangle$. With these values of s (4.11) gives

$$\langle \pm \mid \Phi_{1,2} \mid \pm\rangle = 1.46840...\ (-\lambda)^{\frac{1}{9}} \qquad \langle 0 \mid \Phi_{1,2} \mid 0\rangle = -1.46840...\ \lambda^{\frac{1}{9}}$$
$$\langle \pm \mid \Phi_{1,3} \mid \pm\rangle = 3.70708...\ (-\lambda)^{\frac{2}{3}} \qquad \langle 0 \mid \Phi_{1,3} \mid 0\rangle = 3.70708...\ \lambda^{\frac{2}{3}}$$
$$\langle \pm \mid \Phi_{2,2} \mid \pm\rangle = \pm 1.59427...\ (-\lambda)^{\frac{1}{24}} \qquad \langle 0 \mid \Phi_{2,2} \mid 0\rangle = 0$$
$$\langle \pm \mid \Phi_{2,1} \mid \pm\rangle = \pm 2.45205...\ (-\lambda)^{\frac{35}{72}} \qquad \langle 0 \mid \Phi_{2,1} \mid 0\rangle = 0\ .$$

These numbers can be compared with the numerical results quoted in [7]

$$\begin{aligned}
&\langle \pm \mid \Phi_{1,2} \mid \pm \rangle_{num} = 1.466(6)\ (-\lambda)^{\frac{1}{9}} &&\langle 0 \mid \Phi_{1,2} \mid 0 \rangle_{num} = -1.465(5)\ \lambda^{\frac{1}{9}}\\
&\langle \pm \mid \Phi_{1,3} \mid \pm \rangle_{num} = 3.5(3)\ (-\lambda)^{\frac{2}{3}} &&\langle 0 \mid \Phi_{1,3} \mid 0 \rangle_{num} = 3.4(2)\ \lambda^{\frac{2}{3}}\\
&\langle \pm \mid \Phi_{2,2} \mid \pm \rangle_{num} = \pm 1.594(2)\ (-\lambda)^{\frac{1}{24}} &&\langle 0 \mid \Phi_{2,2} \mid 0 \rangle_{num} = 0\\
&\langle \pm \mid \Phi_{1,2} \mid \pm \rangle_{num} = \pm 2.38(6)\ (-\lambda)^{\frac{35}{72}} &&\langle 0 \mid \Phi_{2,1} \mid 0 \rangle_{num} = 0\ .
\end{aligned}$$

The VEV $\langle\, \Phi_{1,3}\, \rangle$ was earlier estimated in the work [29] as $3.78\ |\lambda|^{\frac{2}{3}}$, with the quoted error of $5-10\%$

5. Minimal models perturbed by the operators $\Phi_{1,5}$

The affine symmetry algebra $U_q(A_2^{(2)})$ of (4.2) contains also the algebra $U_{q^4}(sl_2)$ as a subalgebra. One can use it to obtain another quantum group restriction of (4.2). Let p, p' be two relatively prime integers such that $2p < p'$. As is known, for

$$\beta^2 = \frac{4p}{p'} \tag{5.1}$$

this restriction gives the perturbed minimal model

$$\mathcal{M}_{p/p'} + \bar{\lambda} \int d^2x\, \Phi_{1,5}\ . \tag{5.2}$$

The above condition $2p < p'$ (which excludes unitary models $\mathcal{M}_{p/p+1}$) guarantees that the perturbation is relevant. The coupling parameter $\bar{\lambda}$ in (5.2) is related to the parameters μ, μ' in (4.2) as

$$\bar{\lambda}^2 = \left[\frac{32\,\pi^2\ \mu\ (\mu')^2}{(4-5\beta^2)(1-\beta^2)(4-3\beta^2)(2-\beta^2)} \right]^2 \frac{\Gamma^5\big(1-\frac{\beta^2}{4}\big)\,\Gamma\big(\frac{5\beta^2}{4}\big)}{\Gamma^5\big(\frac{\beta^2}{4}\big)\,\Gamma\big(1-\frac{5\beta^2}{4}\big)} \tag{5.3}$$

According to the general scheme of the quantum group restriction one expects that the restricted theory has a particle of the mass m (possibly among other particles and kinks) given by (3.3) with b^2 replaced by $-\beta^2$ and $\mu \to -\mu,\ \mu' \to -\mu'$. Excluding $\mu\,(\mu')^2$ from these relations, we can expresses the perturbation parameter $\bar{\lambda}$ in terms of the physical mass scale m,

$$m = \frac{2\sqrt{3}\,\Gamma\big(\frac{1}{3}\big)}{\Gamma\big(\frac{3-5\xi}{3-3\xi}\big)\,\Gamma\big(\frac{1+\xi}{3-3\xi}\big)} \left[\frac{4\pi^2\,\bar{\lambda}^2\,(1-4\xi)^2(1-2\xi)^2\,\Gamma^2\big(\frac{3-\xi}{1+\xi}\big)\,\Gamma\big(\frac{\xi}{1+\xi}\big)\,\Gamma\big(\frac{1-4\xi}{1+\xi}\big)}{(1+\xi)^4\ \Gamma^2\big(\frac{4\xi}{1+\xi}\big)\,\Gamma\big(\frac{1}{1+\xi}\big)\,\Gamma\big(\frac{5\xi}{1+\xi}\big)} \right]^{\frac{1+\xi}{12(1-\xi)}}\ . \tag{5.4}$$

Here we use the notation

$$\xi = \frac{p}{p' - p}\ .$$

As it follows from (5.4), QFT (5.2) presumably has a massive spectrum for

$$\begin{aligned} 0 < \xi < \frac{1}{4}\,, &\qquad \Im m\,\bar{\lambda} = 0\ , \\ \frac{1}{4} < \xi < \frac{3}{5}\,, &\qquad \Re e\,\bar{\lambda} = 0\ . \end{aligned} \tag{5.5}$$

Outside this domain the physical content of the model (5.2) is particularly unclear. We restrict our following discussion to the domain (5.5). Then, the specific free energy of (5.2) can be obtained from (3.13) by the substitution $b^2 \to -\frac{4\xi}{1+\xi}$, i.e.

$$f_{1,5} = -\frac{m^2}{16\,\sqrt{3}\,\sin\big(\frac{2\pi\xi}{3-3\xi}\big)\,\sin\big(\frac{\pi(1+\xi)}{3-3\xi}\big)}\ . \tag{5.6}$$

Using this relation and (5.4) one derives the following expression for the expectation value of the perturbing operator in (5.2),

$$\begin{aligned} \langle\,\Phi_{1,5}\,\rangle = & \frac{\pi\,(1-4\xi)(1-2\xi)\,\Gamma\big(\frac{2-2\xi}{1+\xi}\big)}{12\sqrt{3}\,(1+\xi)^2\,\Gamma\big(\frac{4\xi}{1+\xi}\big)\,\sin\big(\frac{2\pi\xi}{3-3\xi}\big)\sin\big(\frac{\pi(1+\xi)}{3-3\xi}\big)}\,\sqrt{\frac{\Gamma\big(\frac{\xi}{1+\xi}\big)\Gamma\big(\frac{1-4\xi}{1+\xi}\big)}{\Gamma\big(\frac{1}{1+\xi}\big)\Gamma\big(\frac{5\xi}{1+\xi}\big)}}\times \\ & \left[\frac{\Gamma\big(\frac{3-5\xi}{3-3\xi}\big)\,\Gamma\big(\frac{1+\xi}{3-3\xi}\big)}{2\,\sqrt{3}\,\Gamma\big(\frac{1}{3}\big)}\right]^{\frac{6\xi-6}{1+\xi}}\ m^{\frac{8\xi-4}{1+\xi}}\ . \end{aligned} \tag{5.7}$$

To obtain more general expectations values, let us note that the generators E_-, H_-, F_- of $U_{q^4}(sl_2)$ commute with the exponential fields $e^{i\frac{k-1}{4}\varphi}(x)$, $k = 1, 2, ...$ which under the restriction become primary fields $\Phi_{1,k}(x)$ in the perturbed theory (5.2). Using the results of Sect.3 and 4 we can obtain

$$\langle 0_s \mid \Phi_{1,k} \mid 0_s\rangle = (-1)^{(k-1)s}\ \left[\frac{m\,(1+\xi)\,\Gamma\big(\frac{3-5\xi}{3-3\xi}\big)\,\Gamma\big(\frac{1+\xi}{3-3\xi}\big)}{2^{\frac{8}{3}}\,\sqrt{3}\,\Gamma\big(\frac{1}{3}\big)}\right]^{2\Delta_{1,k}}\ \mathcal{Q}_{1,5}\big((1+\xi-\xi k\big)\ , \tag{5.8}$$

where $\mathcal{Q}_{1,5}(\eta) = \mathcal{Q}(\eta)/\mathcal{Q}(1)$ and the function $\mathcal{Q}(\eta)$ for $\big|\,\Re e\ \eta\,\big| < \xi$ $(\xi < 1)$ is given by the integral

$$\begin{aligned} \mathcal{Q}(\eta) = \exp\Big\{\int_0^\infty \frac{dt}{t}\,\Big(&\frac{\sinh\big((1-\xi)t\big)\,\cosh\big(2t\eta\big)}{2\sinh\big(3(1-\xi)t\big)\sinh\big((1+\xi)t\big)\sinh(2\xi t)}\times \\ &\Big(\cosh\,\big(3(1-\xi)t\big)+\cosh\,\big((3-\xi)t\big)-\cosh\,\big((1-3\xi)t\big)+\cosh\,\big((1+\xi)t\big)+1\Big)- \\ &\frac{(\eta^2-1)}{2\xi(\xi+1)}\,e^{-4t}\Big)\Big\}\ . \end{aligned}$$

In (5.8), again s is an integer labeling the ground states of (5.2). The ground state structure of (5.2) is not completely understood and we do not discuss it here. Note that the value of s affects only the sign of the expectation values (5.8) with even k.

Some of the perturbed theories (5.2) are studied in the literature [30], [31], [32]. Let us see how our result (5.4) matches the data from these references.

1. The model (5.2) with $(p, p') = (2, 7)$ was studied in [30]. In this case $\Phi_{1,5} = \Phi_{1,2}$. Note that for this value of p there are no kinks in (4.4). However, in this case (5.4) agrees exactly with the mass $m = 2M \sin\left(\frac{\pi}{18}\right)$ (where M is given by (4.9)) of one of the scalar particles of (4.4). In fact, this theory contains two particles with the masses

$$m_1 = \frac{m}{2\cos\left(\frac{\pi}{18}\right)}, \quad m_2 = m \ .$$

The formula (5.4) gives

$$\bar{\lambda} = -i\ 0.0785556...\ m_1^{\frac{18}{7}} \ ,$$

which is in good agreement with the Thermodynamic Bethe Ansatz calculation in [30],

$$\bar{\lambda}_{num} = -i\ 0.0785551...\ m_1^{\frac{18}{7}} \ .$$

2. The model (5.2) with $(p, p') = (2, 9)$ was studied in [31]. The model contains four particles with the masses

$$m_1\,, \quad m_2 = 2m_1 \cos\big(\frac{7\pi}{30}\big), \quad m_3 = 2m_1 \cos\big(\frac{\pi}{15}\big)\,, \quad m_4 = 4m_1 \cos\big(\frac{\pi}{10}\big) \cos\big(\frac{7\pi}{30}\big) \ .$$

Calculations based on the Truncated Conformal Space method [28] in this model give [31]

$$\bar{\lambda}_{num} = -i\ 0.013065...\ m_1^{\frac{10}{3}} \ .$$

Identifying $m = m_2$ we obtain from (5.4)

$$\bar{\lambda} = -i\ 0.0130454...\ m_1^{\frac{10}{3}} \ .$$

3. The case $(p, p') = (3, 14)$ in (5.2) was investigated in [32]. There are six particles with the masses

$$m_1 = m_2\,, \quad m_3 = \sqrt{2} m_1\,, \quad m_4 = m_5 = 2m \cos\big(\frac{\pi}{12}\big)\,, \quad m_6 = 2\sqrt{2}\, m \cos\big(\frac{\pi}{12}\big) \ .$$

The relation

$$\bar{\lambda}_{num} = -i\ 0.011833...\ m_1^{\frac{24}{7}}$$

given in [32] is in good agreement with (5.4) which gives

$$\bar{\lambda} = -i\ 0.01183265...\ m_1^{\frac{24}{7}}\ ,$$

provided we identify $m = m_3$.

It is interesting to notice that in all the above examples the mass m is related to the mass m_1 of the lightest particle in (5.2) as

$$m_1 = \frac{m}{2\sin(\frac{2\pi\xi}{3-3\xi})}\ . \tag{5.9}$$

We believe that this is a general relation for (5.2) which holds as long as $1/5 < \xi < 5/9$.

6. Expectation values of primary fields in the minimal models perturbed by the operator $\Phi_{2,1}$

As is well known, the minimal models $\mathcal{M}_{p/p'}$ admit yet another integrable perturbation

$$\mathcal{M}_{p/p'} + \hat{\lambda} \int d^2x\, \Phi_{2,1}(x)\ . \tag{6.1}$$

The theory (6.1) makes sense for $2p > p'$; this condition guarantees that the operator $\Phi_{2,1}$ is relevant. For $3p > 2p'$ the vacuum structure of (6.1) is expected to be very similar to that of (4.4), with p' playing the role of p. Namely, if p' is odd, i.e. the perturbation is odd with respect to the symmetry (4.12), the theory has $\frac{p'-1}{2}$ degenerate ground states which we denote $\mid 0_s\rangle$ with $s = 1, 3, ..., p'-2$ if $\hat{\lambda} < 0$ and $\mid 0_s\rangle$ with $s = 2, 4, ..., p'-1$ if $\hat{\lambda} > 0$. If p' is even and $\hat{\lambda} > 0$ there are $p'/2 - 1$ ground states $\mid 0_s\rangle$; $s = 2, 4, ..., p'-2$, while if p' is even and $\hat{\lambda} < 0$, there are $p'/2$ vacua $\mid 0_s\rangle$; $s = 1, 3, ..., p'-1$ (compare this with the situation in (4.4) discussed in Sect.4). The excitations are the kinks interpolating between these vacua (and possibly some neutral particles). The natural modification of the conjecture (4.11) in this case is

$$\langle 0_s \mid \Phi_{l,k} \mid 0_s\rangle = \frac{\sin\left(\frac{\pi s}{p'}|p'l - pk|\right)}{\sin\left(\frac{\pi s}{p'}(p'-p)\right)} \left[\frac{M\,\pi\,\xi\,\Gamma\left(\frac{2\xi}{3\xi-3}\right)}{2^{\frac{2}{3}}\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{\xi+1}{3\xi-3}\right)}\right]^{2\Delta_{l,k}} \mathcal{Q}_{2,1}\left((\xi+1)l - \xi k\right)\ , \tag{6.2}$$

where the function

$$\mathcal{Q}_{2,1}(\eta) = \exp\Big\{\int_0^\infty \frac{dt}{t}\ \Big(\frac{\sinh\big((\xi-1)t\big)\ \sinh\big(t(\eta-1)\big)\ \sinh\big(t(\eta+1)\big)}{\sinh\big(3(\xi-1)t\big)\sinh\big((\xi+1)t\big)\sinh(2\xi t)}\times$$
$$\Big(\cosh\big(3(\xi-1)t\big)+\cosh\big((\xi-3)t\big)-\cosh\big((3\xi-1)t\big)+\cosh\big((\xi+1)t\big)+1\Big)- \tag{6.3}$$
$$\frac{(\eta^2-1)}{2\xi(\xi+1)}\,e^{-4t}\Big)\Big\}$$

is obtained from (4.11) by the substitution $\xi \to -1-\xi$, and M is the lightest kink mass. It is remarkable that the function $\mathcal{Q}_{2,1}(\eta)$ coincides with $\mathcal{Q}_{1,5}(\eta)$ in (5.8) continued to the domain $\xi > 2$. For $(l,k)=(2,1)$ (6.3) gives

$$\langle\, \Phi_{2,1}\,\rangle = (-1)^{s-1}\ \frac{2^{2-\frac{7}{2\xi}}\,\xi\,\pi\,\Gamma\big(\frac{3\xi-1}{4\xi}\big)\,\sin\big(\frac{\pi(\xi+1)}{3\xi-3}\big)}{3\sqrt{3}\,(\xi-1)\,\Gamma\big(\frac{\xi+1}{4\xi}\big)\,\sin\big(\frac{2\pi\,\xi}{3\xi-3}\big)}\ \sqrt{\frac{\Gamma\big(\frac12-\frac1\xi\big)}{\Gamma\big(\frac12+\frac1\xi\big)}}\times$$
$$\left[\frac{\sqrt{3}\,\Gamma\big(\frac13\big)\Gamma\big(\frac{\xi+1}{3\xi-3}\big)}{2\pi\,\Gamma\big(\frac{2\xi}{3\xi-3}\big)}\right]^{\frac{3\xi-3}{2\xi}}\ M^{\frac{\xi+3}{2\xi}}\ . \tag{6.4}$$

This formula together with the expression

$$f_{2,1} = -\frac{M^2\ \sin\big(\frac{\pi(\xi+1)}{3\xi-3}\big)}{4\sqrt{3}\ \sin\big(\frac{2\pi\,\xi}{3\xi-3}\big)} \tag{6.5}$$

for the specific free energy of (6.1) leads to the following relation between M and $\hat\lambda$ in (6.1)

$$M = \frac{2^{\frac{\xi-4}{3\xi-3}}\,\sqrt{3}\,\Gamma\big(\frac13\big)\,\Gamma\big(\frac{\xi+1}{3\xi-3}\big)}{\pi\,\Gamma\big(\frac{2\xi}{3\xi-3}\big)}\ \left[\frac{\pi^2\,\hat\lambda^2\,\Gamma^2\big(\frac{3\xi-1}{4\xi}\big)\,\Gamma\big(\frac12-\frac1\xi\big)}{\Gamma^2\big(\frac{\xi+1}{4\xi}\big)\,\Gamma\big(\frac12+\frac1\xi\big)}\right]^{\frac{\xi}{3\xi-3}}\ , \tag{6.6}$$

which reproduces the result of [23].

Some checks of (6.2) can be made. For $(p,p')=(3,4)$ the model (6.1) is just the off-critical Ising field theory with zero magnetic field and $\Phi_{1,2}$ coincides with the order parameter σ. In this case all expectation values (6.2) agree with known result [33],

$$\langle 0_1\mid\ \sigma\ \mid 0_1\rangle = -\langle 0_3\mid\ \sigma\ \mid 0_3\rangle = M^{\frac18}\,2^{\frac{1}{12}}\,e^{-\frac18}\,\mathcal{A}^{\frac32}\,, \qquad \langle 0_2\mid\ \sigma\ \mid 0_2\rangle = 0\ , \tag{6.7}$$

where $\mathcal{A}=1.282427...$ is Glaisher's constant and $M=-2\pi\,\hat\lambda$.

In the case $(p,p')=(4,5)$, (6.1) describes the tricritical Ising model with sub-leading magnetic perturbation. The theory has two degenerate ground states $\mid 0_2\rangle$ and $\mid 0_4\rangle$ (we assume that $\hat\lambda>0$) and the formula (6.2) gives

$$\begin{aligned}\langle 0_4\mid\ \Phi_{2,2}\ \mid 0_4\rangle &= -1.79745...\ \hat\lambda^{\frac{1}{15}}\ , \quad \langle 0_2\mid\ \Phi_{2,2}\ \mid 0_2\rangle = 0.68656...\ \hat\lambda^{\frac{1}{15}}\ ,\\ \langle 0_4\mid\ \Phi_{1,2}\ \mid 0_4\rangle &= 2.04451...\ \hat\lambda^{\frac{8}{45}}\ , \quad \langle 0_2\mid\ \Phi_{1,2}\ \mid 0_2\rangle = -0.78093...\ \hat\lambda^{\frac{8}{45}}\ .\end{aligned} \tag{6.8}$$

In [7] the following numerical estimates for the expectation values of these fields were obtained

$$\begin{aligned} \langle\, \Phi_{2,2}\, \rangle_{num} &\simeq -1.09\ \hat{\lambda}^{\frac{1}{15}}\ , \\ \langle\, \Phi_{1,2}\, \rangle_{num} &\simeq 1.2\ \hat{\lambda}^{\frac{8}{45}}\ . \end{aligned} \tag{6.9}$$

Direct comparison between (6.8) and (6.9) is problematic because it is not clear from [7] precisely which ground state is taken in calculating the expectation values (6.9). The calculations in [7] are based on Truncated Conformal Space method [28] where one starts with the finite-size system with the spatial coordinate compactified on a circle of circumference L; the estimates are then obtained by extrapolating the finite-size results to $L = \infty$. Let us note in this connection that in the finite-size system the above two ground states $\mid 0_2\rangle$ and $\mid 0_4\rangle$ appear in the limit $L \to \infty$ as particular superpositions of two asymptotically degenerate states (with the energy splitting $\sim e^{-ML}$ where M is the kink mass) which we denote $\mid I\rangle$ and $\mid II\rangle$ (the first being the true ground state at finite L). Simple calculation which takes into account the known kink picture of the excitations in this theory (see [10], [34]) gives the following exact relation between these states at $L = \infty$

$$\begin{aligned} \mid I\rangle &= \frac{1}{2\cos\left(\frac{\pi}{10}\right)}\ \mid 0_4\rangle + \frac{\cos\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{10}\right)}\ \mid 0_2\rangle\ , \\ \mid II\rangle &= \frac{2\cos\left(\frac{\pi}{10}\right)}{\sqrt{5}}\ \mid 0_4\rangle - \frac{\cos\left(\frac{\pi}{10}\right)}{\sqrt{5}\cos\left(\frac{\pi}{5}\right)}\ \mid 0_2\rangle\ . \end{aligned} \tag{6.10}$$

It is easy to check using (6.2) that

$$\langle I \mid \Phi_{2,2} \mid I\rangle = \langle I \mid \Phi_{1,2} \mid I\rangle = 0\ . \tag{6.11}$$

This result is not at all surprising as at finite L the ground state $\mid I\rangle$ is obtained by perturbing the conformal ground state (i.e. the primary state $\mid \Phi_{1,1}\rangle$) by the operator $\Phi_{2,1}$. Therefore at any finite L the state $\mid I\rangle$ is a superposition of the states out of the conformal families $\big[\,\Phi_{1,1}\big], \big[\,\Phi_{2,1}\big]$ and $\big[\,\Phi_{3,1}\big]$. By the conformal fusion rules all these states produce vanishing expectation values of both $\Phi_{1,2}$ and $\Phi_{2,2}$. In fact, the result (6.11) can be considered as a nontrivial consistency check of our conjecture (6.2). On the other hand, the expectation values

$$\begin{aligned} \langle II \mid \Phi_{2,2} \mid II\rangle &= -\langle II \mid \Phi_{2,2} \mid I\rangle = 1.11089...\ \hat{\lambda}^{\frac{1}{15}}\ , \\ \langle II \mid \Phi_{1,2} \mid II\rangle &= \langle II \mid \Phi_{1,2} \mid I\rangle = 1.26358...\ \hat{\lambda}^{\frac{8}{45}}\ , \end{aligned} \tag{6.12}$$

do not vanish and actually the numerical results (6.9) appear to be reasonably close to $\langle II \mid \Phi_{2,2} \mid I\rangle$ and $\langle II \mid \Phi_{1,2} \mid I\rangle$. We believe therefore that it is these expectation values that are quoted in [7].

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